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EXACT SOLUTIONS OF THE AXIALLY SYMMETRIC EULER EQUATIONS†

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New classes of exact solutions of Euler's equations are found, which describe steady axially symmetric flow with a vortex. Examples of solutions corresponding to fluid flows with a free boundary are given. Copyright © 1996 Elsevier Science Ltd.

1. We consider the following equation for the stream function $\psi(z, r)$ describing steady axially symmetric (with vortex) flows of an ideal incompressible fluid [1], which is also known as the Grad–Shafranov equation in plasma physics

$$\psi_{zz} + \psi_{rr} - r^{-1}\psi_r = r^2 F - H \tag{1.1}$$

where F and H are arbitrary functions of ψ. We will seek solutions ψ of (1.1) such that for any smooth function ϕ the composition $\chi = \phi \circ \psi$ is also a solution of some equation

$$\chi_{zz} + \chi_{rr} - r^{-1}\chi_r = r^2 F_1(\chi) - H_1(\chi) \tag{1.2}$$

Solutions having this property will be called functionally-invariant. Functionally-invariant solutions of the wave equation were constructed in [2] and their group-theoretic interpretation was given in [3].

Henceforth we shall assume that $\psi_z \neq 0$ and $\psi_r \neq 0$. Substituting $\phi \circ \psi$ in place of χ in (1.2), we obtain

$$\phi'(\psi)(\psi_{zz} + \psi_{rr} - r^{-1}\psi_r) + \phi''(\psi)(\psi_z^2 + \psi_r^2) = r^2 F_1(\phi(\psi)) - H_1(\phi(\psi)) \tag{1.3}$$

If $\phi'' \neq 0$, then from (1.1) and (1.3) we obtain the equation

$$\psi_z^2 + \psi_r^2 = r^2 C_2 + C_1 \tag{1.4}$$

$$C_1(\psi) = -(H_1(\phi(\psi)) - \phi'(\psi)H(\psi)) / \phi''(\psi)$$

$$C_2(\psi) = (F_1(\phi(\psi)) - \phi'(\psi)F(\psi)) / \phi''(\psi)$$

Multiplying (1.1) by ψ_z and subtracting from it Eq. (1.4) differentiated with respect to z and multiplied by two, we arrive at the equation

$$r\psi_z(\psi_r / r)_r - r\psi_r(\psi_r / r)_z = (A_2(\psi)r^2 + A_3(\psi))\psi_z \tag{1.5}$$

$$A_2(\psi) = F - C_2' / 2, \quad A_3(\psi) = -(H + C_1' / 2)$$

Integrating the latter, we get

$$\psi_r = A_1(\psi)r + A_2(\psi)r^3 / 2 + A_3(\psi)r \ln r \tag{1.6}$$

The compatibility of (1.4) and (1.6) can be investigated by standard methods. Expressing ψ_z in (1.4) in terms of ψ and r using (1.6), after cross-differentiation we obtain

$$K_1 r \ln^2 r + K_2 r^3 \ln r + K_3 r \ln r + K_4 r^5 + K_5 r^3 + K_6 r = 0$$

$$K_1 = -2A_3^2, \quad K_2 = -8A_3 A_2 + C_2' A_3 - 2C_2 A_3'$$

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$$K_3=-4A_1A_3+C_1'A_3-2C_1A_3',\quad K_4=-6A_2^2+C_2'A_2-2C_2A_2' \tag{1.7}$$
$$K_5=-8A_1A_2-2A_2A_3+C_1'A_2+C_2'A_1-2C_1A_2'-2C_2A_1'$$
$$K_6=2C_2-2A_1A_3-2A_1^2+C_1'A_1-2C_1A_2'$$

Since $\psi_z \neq 0$, Eq. (1.7) implies the six equations $K_i = 0$. By the first equation, $H + C'_1/2 = 0$. Then the second and third equations become identities, and the fourth to sixth equations take the form

$$\begin{gathered} -6A_2^2+C_2'A_2-2C_2A_2'=0 \\ -8A_1A_2+C_1'A_2+C_2'A_1-2C_1A_2'-2C_2A_1'=0 \\ 2C_2-2A_1^2+C_1'A_1-2C_1A_2'=0 \end{gathered} \tag{1.8}$$

Using the above, we can write (1.6) as follows:

$$\psi_r=A_1(\psi)r+A_2(\psi)r^3 \tag{1.9}$$

and the functions F and H can be expressed by

$$F=C_2'/2+2A_2,\quad H=-C_1'/2 \tag{1.10}$$

It is obvious that every solution of Eqs (1.4) and (1.9) is functionally-invariant. It can be shown that Eq. (1.1), taking (1.10) into account, is satisfied identically by (1.4) and (1.9). Therefore we have proved the following result.

Theorem. The function $\psi(z, r)$ is a functionally-invariant solution of Eq. (1.1) if and only if it satisfies Eqs (1.4) and (1.9), where A_i, C_i $(i = 1, 2)$ are solutions of (1.8).

2. Among the functionally-invariant solutions one can distinguish those corresponding to fluid flow with free boundary. If the pressure is assumed to depend only on the stream functions $p = p(\psi)$ it can be shown that for any constant ψ_0 the conditions for the existence of free boundary [4] will be satisfied for the surface $\psi = \psi_0$. Using Bernoulli's equation

$$p+|\mathbf{u}|^2/2=R(\psi)$$

for an incompressible fluid in the axially symmetric case, we arrive at the conclusion that the square of the velocity depends on the stream function $|\mathbf{u}|^2 = T(\psi)$, which is in turn equivalent to Eq. (1.4) for $C_1 \leqslant 0$. It follows that for any functionally-invariant solution ψ of Eq. (1.1) the surface $\psi(z, r) = \psi_0 =$ const can serve as a free boundary subject to the condition $C_1(\psi_0) \leqslant 0$.

3. The problem arises of finding the solutions of system (1.8). First we will consider the case $A_2 = 0$. Then, the first equation in (1.8) is an identity. Without loss of generality we can set $A_1 = 1$. The solutions of Eq. (1.1) corresponding to other values of A_1 can be obtained as a result of the superposition of a function ϕ and the solution obtained for $A_1 = 1$. The second and third equations (1.8) can be simplified considerably.

$$C_2'=0,\quad 2C_2-2+C_1'=0$$

Their solution is $C_2 = k$, $C_1 = l + (2 - 2k)\psi$, where $k, l, \in R$. Substituting the values just found into the right-hand sides of (1.4) and (1.9) and solving the resulting overdetermined system, we obtain

$$\psi=\begin{cases} r^2/2+\sqrt{l}z+\gamma, & k=1, l\geqslant 0 \\ r^2/2+(1-k)(z+\gamma)/2+l/(2-2k), & k\neq 1 \end{cases} \tag{3.1}$$

where γ is an arbitrary constant. In the case when $k > 1$ and $\psi_0 \geqslant l/(2k - 2)$ the surface $\psi = \psi_0$ (which is a one-sheeted hyperboloid if the second inequality is strict, and a circular cone otherwise) can be regarded as a free boundary. As has been mentioned above, $\phi \circ \psi$, where ψ is given by (3.1), is a solution of Eq. (1.1). Moreover

$$F = -k\tau''/\tau'^3, \quad H = l\tau''/\tau'^3 + (1-k)\tau''\tau/\tau'^3 + (k-1)/\tau'$$

Here ϕ is an arbitrary function and τ is its inverse.

We will now consider the case when $A_2 \neq 0$. We make the substitution

$$A_1 = BA_2, \quad C_1 = (K+B)EA_2^2, \quad C_2 = EA_2^2$$

where $K(\psi), E(\psi), B(\psi)$ are new unknown functions. Then system (1.8) will be equivalent to the following

$$A_2E' = 6, \quad A_2EK' = E/K - 4B - 6K, \quad A_2KB' = 1 \tag{3.2}$$

Without loss of generality, we can set $A_2 = \psi^3$ in (3.2). Then the resulting system admits of the expansion operator $X = 2E\partial_E + K\partial_K + B\partial_B - \psi\partial_\psi$, the solution invariant under this operator being given by

$$E = -3\psi^{-2}, \quad B = \pm 3\psi^{-1}, \quad K = \mp\psi^{-1}/3$$

It can be verified that there are no real solutions when $B = 3\psi^{-1}$ and $K = -\psi^{-1}/3$, so that in this case (1.4) implies that $\psi_z^2 + \psi_r^2 < 0$. When $B = -3\psi^{-1}$, $K = -\psi^{-1}/3$, from (1.4) and (1.9) we obtain the following system from which to find ψ

$$\psi_z^2 + \psi_r^2 = -3\psi^4 r^2 + 8\psi^3, \quad \psi_r = \psi^3 r^3 - 3\psi^2 r$$

This system admits of the operator $Y = z\partial_z + r\partial_r - 2\psi\partial_\psi$. The solutions invariant under Y are given by

$$\psi = \frac{1}{r^2}\left(1 \pm \frac{z}{\sqrt{r^2+z^2}}\right) \tag{3.3}$$

The choice of sign obviously does not matter. Other solutions of the system are either trivial (i.e. independent of z) or can be obtained from (3.3) by a translation along the z axis.

The function $\phi \circ \psi$ is a solution of Eq. (1.1) when

$$F = -4\tau^3/\tau' + 3\tau^4\tau''/\tau'^3, \quad H = 12\tau^2/\tau' - 8\tau^3\tau''/\tau'^3$$

Here, as before, $\tau(\psi)$ is the inverse function to ϕ.

It should be noted that solutions of the form $\phi \circ \psi$ for $\phi = 2\ln\psi + \delta$ and $\phi = \varepsilon\psi^{2/(n-1)}$, where $\varepsilon, \delta \in R$, are invariant solutions with respect to the admissible operators $z\partial_z + r\partial_r - 4\partial_\psi$ and, respectively, $z\partial_z + r\partial_r - 4(n-1)^{-1}\psi\partial_\psi$, which were found for the first time in [5].

4. Apart from those described above, we have not succeeded in finding other explicit representations of functionally-invariant solutions of Eq. (1.1). Nevertheless, the level curves of ψ corresponding to functionally-invariant solutions can be constructed numerically. In this way a picture of the behaviour of trajectories of the corresponding solutions can be built up. One does not have to solve (1.8) to construct a separate level curve; it suffices to use the equation

$$dz/dr = -\psi_r/\psi_z$$

When constructing the level curves the initial points were varied along the line $z = z_0 = \text{const}$. Along with solving Eq. (1.9), it turned out to be necessary to solve system (1.8) simultaneously.

Below we present some computational results. To fix our ideas we will put $E = \psi$ everywhere. Then the first equation in (3.2) implies that $A_2 = 6$. In addition, the initial point for constructing the level curve was chosen to be on the Oz axis. When constructing the level curves shown in Fig. 1, we selected the initial data to be $r = 2$, $B = -1.98$, $E = 8.17$, $K = -1.99 \times 10^{-2}$. The values $\psi = 8.17; 11.1; 17.0; 29.1$ of the stream function correspond to curves 1–4. For each of the level curves shown $\psi = \psi_0 C_1(\psi_0) \leq 0$. It follows that with each of them one can associate a surface which is a free boundary. The streamlines are obviously open. We used the following initial data for the level curves in Fig. 2: $r = 2.829$, $B = -5.84$, $E = 5.98$, $K = 4.10$. For curves 1–5 $\psi = 5.98; 34.6; 44.2; 53.7; 82.5$, respectively. For each level curve $\psi = \psi_0$ that does not intersect the axis of rotation the condition $C_1(\psi_0) \leq 0$ is also satisfied. The resulting solution can be interpreted as the motion of a fluid with toroidal free boundary.

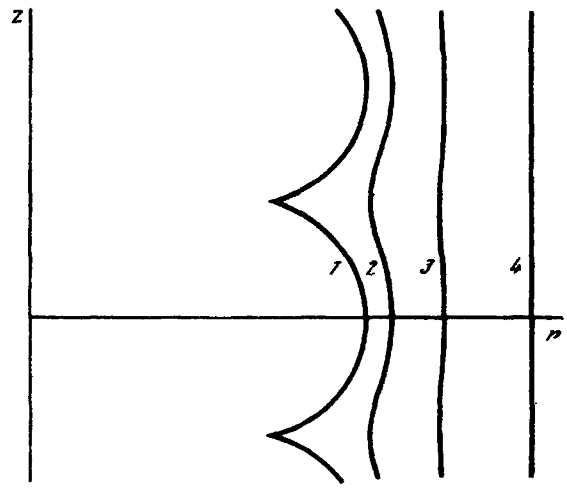

Fig. 1.

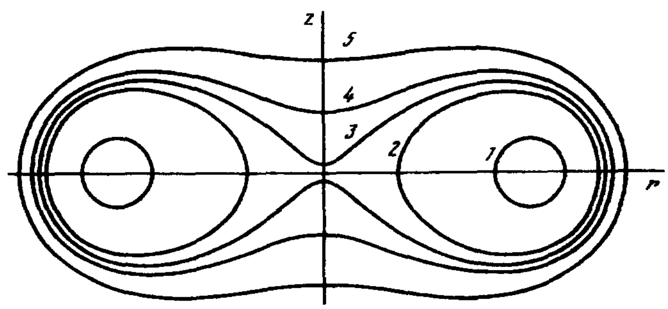

Fig. 2.

5. Let us consider a more general form of Eqs (1.4) and (1.9)

$$\psi_r = A(\psi, r), \quad \psi_z^2 + \psi_r^2 = C(\psi, r) \tag{5.1}$$

The compatibility condition for these equations is given by

$$C_r - 2AA_r + C_\psi A - 2CA_\psi = 0 \tag{5.2}$$

Equation (1.1) will be satisfied by virtue of (5.1) if and only if

$$C_\psi / 2 + A_r - A / r = Fr^2 - H \tag{5.3}$$

Equations (5.1)–(5.3) can be regarded as a group stratification [6] of Eq. (1.1) relative to the one-parameter group generated by the operator ϑ_z, Eqs (5.1) forming an automorphic system and Eqs (5.2) and (5.3) forming a resolving system.

We shall seek a solution of system (5.2), (5.3) in the form

$$A = \sum_{i=1}^{m} A_i(\psi)S_i(r), \quad C = \sum_{j=1}^{n} C_j(\psi)T_j(r)$$

where m and n are certain natural numbers. The case $m = n = 2$ corresponds to functionally-invariant solutions. It can be shown that the case $m = 1, n = 3$ corresponds to solutions of the form $\psi = f(z)g(r)$

when $F = A\psi$ and $H = B\psi \ln \psi$ [5]. It turns out that for $m = 1$ and $n = 3$ other solutions of Eqs (5.2), (5.3) of the given form exist.

For example, let $S_1 = r^{-1}$, $T_1 = r^{-1}$, $T_2 = 1$, $T_3 = r^2$. Substituting the above representations for A and C into Eqs (5.2) and (5.3), we obtain the equations

$$(-2C_1 + 2A_1^2 + C_1'A_1 - 2C_1A_1')r^{-3} + (C_2'A_1 - 2C_2A_1')r^{-1} + (2C_3 + C_3'A_1 - 2C_3A_1')r = 0$$
$$(C_1'/2 - 2A_1)r^{-2} + C_2'/2 + C_3'/2r^2 = Fr^2 - H$$

Equating the coefficients of like powers of r and solving the resulting system of ordinary differential equations, we find that

$$C_1 = (k\psi^2 - 1)^2/k, \quad C_2 = \beta\psi^2(k\psi^2 - 1)^2$$
$$C_3 = \alpha\psi^4(k\psi^2 - 1), \quad A_1 = \psi(k\psi^2 - 1)$$
$$F = \alpha\psi^3(3k\psi^2 - 2), \quad H = \beta(k\psi^2 - 1)(\psi - 3k\psi^3); \quad \alpha, \beta, k \in R$$

The function ψ is specified, apart from an arbitrary additive constant. Besides, no solution corresponding to a linear equation of the form (1.1) is given. The solution of Eq. (1.1) for the above A and C is given by

$$\psi = (k(1 + Mr^2))^{-1/2}$$

where M is the Weierstrass function, satisfying the equation

$$M_z^2 = 4M^3 + 4\beta M^2 - 4\alpha M/k$$

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